The Helly number of Hamming balls and related problems

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Abstract

We prove the following variant of Helly's classical theorem for Hamming balls with a bounded radius. For $n > t$ and any (finite or infinite) set X, if in a family of Hamming balls of radius t in X^n, every subfamily of at most 2^{t+1} balls have a common point, so do all members of the family. This is tight for all $|X| > 1$ and all $n > t$. The proof of the main result is based on a novel variant of the so-called dimension argument, which allows one to prove upper bounds that do not depend on the dimension of the ambient space. We also discuss several related questions and connections to problems and results in extremal finite set theory and graph theory.

1 Introduction

Helly's theorem, proved by Helly more than 100 years ago ([Hel23]), is a fundamental result in Discrete Geometry. It asserts that a finite family of convex sets in the d-dimensional Euclidean space has a nonempty intersection if every subfamily of at most $d + 1$ of the sets has a nonempty intersection.

This theorem, in which the number $d+1$ is tight, led to numerous fascinating variants and extensions in geometry and beyond (c.f., e.g., [Eck93], [BK22] for two survey articles). It motivated the definition of the *Helly number* $h(\mathcal{F})$ for a general family $\mathcal{F}$ of sets. This is the smallest integer h such that for any finite subfamily $\mathcal{K}$ of $\mathcal{F}$, if every subset of at most h members of $\mathcal{K}$ has a nonempty intersection then all sets in $\mathcal{K}$ have a nonempty intersection. The classical theorem of Helly asserts that the Helly number of the family of convex sets in $\mathbb{R}^d$ is $d + 1$. An additional example of a known Helly number is the Theorem of Doignon [Doi73] that asserts that the Helly number of convex lattice sets in d-space, that is sets of the form $C \cap Z^d$ where C is a convex set in $\mathbb{R}^d$, is 2^d. A more combinatorial example is the fact that the Helly number of the collection of (sets of vertices of) subtrees of any tree is 2.

In the spaces X^n for finite or infinite X, the *Hamming balls* are among the most natural objects to study. The *Hamming distance* between $p, q \in X^n$, denoted by $\mathrm{dist}(p, q)$, is the number of coordinates where p and q differ, and the Hamming ball of radius t centered at $x \in X^n$, denoted by $B(x, t)$, is the set of all points $p \in X^n$ that satisfy $\mathrm{dist}(p, x) \leq t$. Note that every Hamming ball of radius t is the whole space if $n \leq t$. Hence, we may and will always assume that $n \geq t + 1$. Our main result in the present paper is the determination of the Helly number of the family of all Hamming balls of radius t in the space X^n, where X is an arbitrary (finite or infinite) set.

Theorem 1.1. *Let $n > t \geq 0$ and X be any set of cardinality $|X| \geq 2$. The Helly number $h(n, t; X)$ of the family of all Hamming balls of radius t in X^n is exactly 2^{t+1}.*

Crucially, $h(n, t; X)$ depends only on t. We note that the special case $X = \{0, 1\}$ of this theorem settles a recent problem raised in [RST23], where the question is motivated by an application in

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learning theory. See also [BHMZ20] for more about the connection between Helly numbers and questions in computational learning. The proof further shows that the Helly number of the family of all Hamming balls of radius at most t in X^n is also 2^{t+1}.

The proof of the main result is based on a novel variant of the so-called dimension argument. Surprisingly, this variant allows us to prove some upper bounds that do not depend on the dimension of the ambient space. We believe that this may have further applications. For the special case of binary strings, that is, $|X| = 2$, we prove a stronger statement by a probabilistic argument. For convenience, we define the following two functions $f(t, X)$ and $f'(t, X)$.

Definition 1.2. *Let $t \geq 0$ and X be any set of cardinality $|X| \geq 2$. Define*

- *$f(t; X)$ to be the maximum m such that there exists $n > t$ and $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_m \in X^n$ where dist$(a_i, b_i) \geq t + 1$ for all $i \in [m]$ and dist$(a_i, b_j) \leq t$ for all distinct $i, j \in [m]$;*
- *$f'(t; X)$ to be the maximum m such that there exists $n > t$ and $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_m \in X^n$ where dist$(a_i, b_i) \geq t+1$ for all $i \in [m]$ and dist$(a_i, b_j) +$ dist$(a_j, b_i) \leq 2t$ for all distinct $i, j \in [m]$.*

The study of these functions can also be motivated by the well-known set-pair inequalities in extremal set theory. The set-pair inequalities, initiated by Bollobás [Bol65], play an important role in extremal combinatorics with applications in the study of saturated (hyper)-graphs, τ-critical hypergraphs, matching-critical hypergraphs, and more. See [Tuz94, Tuz96] for surveys. A significant generalization of Bollobás' result is due to Füredi [Für84]. It states that if $A_1, A_2, \ldots, A_m$ are sets of size a and $B_1, B_2, \ldots, B_m$ are sets of size b such that $|A_i \cap B_i| \leq k$ for all $i \in [m]$ and $|A_i \cap B_j| > k$ for $1 \leq i < j \leq m$, then $m \leq \binom{a+b-2k}{a-k}$, and this is tight. Using this result, one can give a short argument that $f(t; X)$ is finite.

Proposition 1.3. *$f(t; X) \leq \binom{2t+2}{t+1}$ for every $t \geq 0$ and every set X.*

Proof. Suppose, for some $n > t$, that there exist $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_m \in X^n$ satisfying $\mathrm{dist}(a_i, b_i) \geq t + 1$ for all i and $\mathrm{dist}(a_i, b_j) \leq t$ for all distinct $i, j \in [m]$. For each $i \in [m]$, let $A_i := \{(k, a_k) : k = 1, 2, \ldots, n\}$ and $B_i := \{(k, b_k) : k = 1, 2, \ldots, n\}$ be sets of n pairs. Then $|A_i \cap B_j| + \mathrm{dist}(a_i, b_j) = n$. Thus, $|A_i \cap B_i| \leq n - t - 1$ for all $i \in [m]$ and $|A_i \cap B_j| \geq n - t$ for all distinct $i, j \in [m]$. Since $|A_i| = |B_i| = n$, the above result of Füredi implies $m \leq \binom{2n-2(n-t-1)}{n-(n-t-1)} = \binom{2t+2}{t+1}$, as desired. ■

We note that $h(n, t; X) \leq f(t; X) \leq f'(t; X)$ whenever $n > t$. To see the first inequality, let $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$ be a minimal collection of Hamming balls of radius t in X^n such that any $h(n, t; X) - 1$ of them intersect, but all of them do not (the existence comes from the minimality of $h(n, t; X)$). This means $m \geq h(n, t; X)$ and every $m - 1$ balls intersect (using that $\mathcal{B}$ is minimal). For each $i \in [m]$, let a_i be the center of B_i and there exists $b_i \in \bigcap_{j \neq i} B_j \setminus B_i$. Then $\mathrm{dist}(a_i, b_i) \geq t + 1$ for all $i \in [m]$ and $\mathrm{dist}(a_i, b_j) \leq t$ for all distinct $i, j \in [m]$. Thus, $h(n, t; X) \leq m \leq f(t; X)$. The second inequality is a direct consequence of Definition 1.2.

For general X, the Helly-type result for Hamming balls (Theorem 1.1) follows from the following.

Theorem 1.4. *$f(t; X) = 2^{t+1}$ for every $t \geq 0$ and every set X with $|X| \geq 2$.*

In the binary case, we further prove the following.

Theorem 1.5. *$f'(t; \{0, 1\}) = 2^{t+1}$ for every $t \geq 0$.*

Indeed, our proof of Theorems 1.4 and 1.5 works in the more general setting where we assume $\mathrm{dist}(a_i, b_i) \geq t + s$ (for some $s \geq 1$) instead of $\mathrm{dist}(a_i, b_i) \geq t + 1$. For simplicity, we denote $f(t, s; X)$ and $f'(t, s; X)$ as the corresponding families' largest size. Precisely, our proof shows that $f(t, s; X) \leq 2^{t+s}/V_{t+s,s}$ and $f'(t, s; \{0, 1\}) \leq 2^{t+s}/V_{t+s,s}$, where

$$V_{n,d} := \begin{cases} \sum_{i=0}^{(d-1)/2} \binom{n}{i} & d \text{ is odd} \\ \sum_{i=0}^{d/2-1} \binom{n}{i} + \binom{n-1}{d/2-1} & d \text{ is even} \end{cases} \tag{1}$$

We note that $V_{n,d}$ is the size of the Hamming ball in $\{0, 1\}^n$ of radius $\frac{d-1}{2}$ if d is odd and of the union of two Hamming balls in $\{0, 1\}^n$ of radius $\frac{d}{2} - 1$ whose centers are of Hamming distance 1 if d is even. Interestingly, $V_{n,d}$ is also known to be the maximum possible cardinality of a set of points of diameter at most $d - 1$ in $\{0, 1\}^n$; see [Kat64, Kle66, Bez87]. We also note that when d is odd, $2^n/V_{n,d}$ is the well-known Hamming bound for the maximum possible number of codewords in a binary *error correcting code* (ECC) of length n and distance d. Binary ECCs, which are large collections of binary strings with a prescribed minimum Hamming distance between any pair, are widely studied and applied in computing, telecommunication, information theory and more; see [MS77a, MS77b]. Indeed, ECCs naturally define a_is and b_is in Definition 1.2. As we will show in Section 3, the existence of ECCs that match the Hamming bound (the so-called *perfect codes*) and their extensions imply that $f(t, s; X) = f'(t, s; \{0, 1\}) = 2^{t+s}/V_{t+s,s}$ when $s \in \{1, 2\}$, or $s \in \{3, 4\}$ and $t + 4$ is a power of 2, or $s \in \{7, 8\}$ and $t = 16$. This will be shown using the well-known Hamming code and the Golay code. In addition, the famous BCH codes discovered by Bose, Chaudhuri and Hocquenghem imply that our bounds are close to being tight when s is fixed, that is, $f(t, s; X) = \Theta_s(2^{t+s}/V_{t+s,s})$ and $f'(t, s; X) = \Theta_s(2^{t+s}/V_{t+s,s})$.

Another well-known result in extremal set theory due to Tuza [Tuz87] states that if $(A_i, B_i)_{i=1}^m$ satisfies $A_i \cap B_i = \emptyset$ for $i \in [m]$ and $(A_i \cap B_j) \cup (A_j \cap B_i) \neq \emptyset$ for distinct $i, j \in [m]$, then $\sum_{i=1}^m p^{|A_i|}(1 - p)^{|B_i|} \leq 1$ for all $0 < p < 1$. This also has various applications; see [Tuz94, Tuz96]. When $|A_i| + |B_i| = t + 1$ for all i, this result implies $m \leq 2^{t+1}$, which is tight. Theorem 1.5 generalizes this by taking $A_i := \{k \in [n] : a_{i,k} = 1\}$ and $B_i := \{k \in [n] : b_{i,k} = 1\}$: if $|A_i \triangle B_i| \geq t + 1$[1] for all $i \in [m]$ and $|A_i \triangle B_j| + |A_j \triangle B_i| \leq 2t$ for all distinct $i, j \in [m]$, then $m \leq f'(n, t, \{0, 1\}) = 2^{t+1}$. Here, we do not require A_i and B_i to be disjoint.

Recall that given a finite family of convex sets in $\mathbb{R}^d$, Helly's theorem asserts that if any collection of at most $(d + 1)$ sets in the family intersect, then all intersect. The following two extensions of Helly's theorem received a considerable amount of attention. The fractional Helly theorem, first proved by Katchalski and Liu [KL79], states that if an α-fraction of $(d+1)$-tuples of sets in the family intersect, then one can select a β-fraction of the sets in the family with a nonempty intersection. The Hadwiger-Debrunner conjecture, also known as the (p, q)-theorem, was first proved by Alon and Kleitman [AK92]. It states that for $p \geq q \geq d + 1$, if among any p sets in the family, q of them intersect, then there is a set of $O_{d,p,q}(1)$ points in $\mathbb{R}^d$ such that every set in the family contains at least one of these points. See also [BK22] for more recent variants and extensions. We consider these two types of problems for Hamming balls of fixed radius. Interestingly, for both of them, if $|X| = 2$, e.g. binary strings, we only need the information on pairs of balls; if $|X| = \infty$, we need the information on $t + 2$ balls.

Finally, we mention briefly that Theorem 1.4 motivates the study of a natural variant of the *Prague dimension* (also called the *product dimension*) of graphs. Initiated by Nešetřil, Pultr and Rödl [NP77, NR78], the Prague dimension of a graph is the minimum d such that every vertex is assigned to a unique

[1]We write $A \triangle B := (A \setminus B) \cup (B \setminus A)$ for the symmetric difference of A and B.

vector in $\mathbb{Z}^d$ and two vertices are connected by an edge if and only if the corresponding vectors differ in all coordinates, i.e, it is the minimum possible number of proper vertex colorings of G so that for every pair u, v of non-adjacent vertices there is at least one coloring in which u and v have the same color. This notion has been studied intensively, see, e.g., [LNP80, Alo86, ER96, Für00, AA20, GPW23].

The rest of this paper is organized as follows. In Section 2 we present the proof of the main result Theorem 1.4. Section 3 deals with binary strings and briefly discusses the behavior of $f'(n, t, X)$ for $|X| > 2$. In Section 4 we discuss several variants and generalizations of the main results including a fractional Helly-type result, a Hadwiger-Debrunner-type result, a variant of the Prague dimension of graphs, and a generalization of $f(t; X)$ to sequences of sets. The final Section 5 contains some concluding remarks and open problems.

2 General strings

We start by describing the lower bound of Theorem 1.1, given by [RST23]. This also provides the lower bounds in Theorems 1.4 and 1.5 as $f'(t; X) \geq f(t; X) \geq h(n, t; X) \geq 2^{t+1}$

Proposition 2.1. *$h(n, t; X) \geq 2^{t+1}$ for any $n > t$ and any set X of cardinality $|X| \geq 2$.*

Proof. We may assume $n = t + 1$ as $h(n, t; X) \geq h(t + 1, t; X)$ and that $0, 1 \in X$. Consider all the 2^{t+1} Hamming balls $B(a, t)$ where $a \in \{0, 1\}^{t+1}$. It suffices to show that any $2^{t+1} - 1$ balls intersect while all of them do not. Observe that $B(a, t) = \{0, 1\}^{t+1} \setminus \{\overline{a}\}$ where $\overline{a} \in \{0, 1\}^{t+1}$ is given by flipping all coordinates of a, i.e. changing 0 to 1 and changing 1 to 0. Therefore, for any $\ell = 2^{t+1} - 1$ vectors $a_1, \ldots, a_\ell \in \{0, 1\}^{t+1}$, the intersection $\bigcap_{i=1}^{\ell} B(a_i, t)$ contains all but at most $\ell < 2^{t+1}$ elements in $\{0, 1\}^{t+1}$. In other words, any $2^{t+1} - 1$ such Hamming balls intersect. On the other hand, $\bigcap_{a\in\{0,1\}^{t+1}} B(a, t) = \bigcap_{a\in\{0,1\}^{t+1}}(\{0, 1\}^{t+1} \setminus \{\overline{a}\}) = \{0, 1\}^{t+1} \setminus \bigcup_{a\in\{0,1\}^{t+1}}\{\overline{a}\} = \emptyset$, i.e. all the 2^{t+1} balls do not intersect. ■

The rest of this section contains the proof of the upper bound of Theorem 1.4, and thus also of Theorem 1.1. To this end, we need the following properties of $V_{n,d}$.

Claim 2.2. *$V_{n,d} \geq 2V_{n-1,d-1}$ for $2 \leq d \leq n$ and this is an equality if d is even. In particular, $V_{n,d} \geq 2^{d-1}$ for all $1 \leq d \leq n$.*

Proof. If $d = 2k$ for some $k \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$, then

$$V_{n,d} = \sum_{i=0}^{k-1} \binom{n}{i} + \binom{n-1}{k-1} = \sum_{i=0}^{k-1} \binom{n-1}{i} + \sum_{i=0}^{k-2} \binom{n-1}{i} + \binom{n-1}{k-1} = 2\sum_{i=0}^{k-1} \binom{n-1}{i} = 2V_{n-1,d-1}.$$

If $d = 2k - 1$ for some $k \in \{2, 3, \ldots, \lfloor \frac{n+1}{2} \rfloor\}$, then

$$V_{n,d} = \sum_{i=0}^{k-1} \binom{n}{i} = 2\sum_{i=0}^{k-2} \binom{n-1}{i} + \binom{n-1}{k-1} \geq 2\sum_{i=0}^{k-2} \binom{n-1}{i} + 2\binom{n-2}{k-2} = 2V_{n-1,d-1}.$$

Here, we used $\binom{n-1}{k-1} \geq 2\binom{n-2}{k-2}$ as $k \leq \frac{n+1}{2}$.

Given the first inequality, $V_{n,d} \geq 2V_{n-1,d-1} \geq \cdots \geq 2^{d-1}V_{n-d+1,1} = 2^{d-1}$, as desired. ■

We now show the upper bound of Theorem 1.4 by the following stronger theorem.

Theorem 2.3. *Let $n > t \geq 0, m \geq 1$, and X be nonempty. Suppose $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_m \in X^n$, and assume that for each $i \in [m]$, dist$(a_i, b_i) = t + s_i$ for some $s_i \geq 1$, and dist$(a_i, b_j) \leq t$ for all distinct $i, j \in [m]$. Then,*

$$\sum_{i=1}^{m} \frac{V_{t+s_i,s_i}}{2^{t+s_i}} \leq 1. \tag{2}$$

In particular, $f(t; X) \leq 2^{t+1}$ and $f(t, s; X) \leq 2^{t+s}/V_{t+s,s}$ if $s_i \geq s$ for all $i \in [m]$.

Proof. First, suppose we have proved Eq. (2). Then, Claim 2.2 implies $1 \geq m \cdot 2^{s_i-1}/2^{t+s_i} \geq m/2^{t+1}$, i.e. $m \leq 2^{t+1}$. Hence, $f(t; X) \leq 2^{t+1}$. Similarly, if $s_i \geq s$ for all $i \in [m]$, using Claim 2.2, we acquire $1 \geq \sum_{i=1}^{m} \frac{V_{t+s_i,s_i}}{2^{t+s_i}} \geq m \cdot V_{t+s,s}/2^{t+s}$, i.e. $m \leq 2^{t+s}/V_{t+s,s}$. So, $f(t, s; X) \leq 2^{t+s}/V_{t+s,s}$.

In the rest of the proof, we establish Eq. (2). The proof is algebraic and uses a novel variant of the dimension argument which provides a dimension-free upper bound. Without loss of generality, assume $X \subseteq \mathbb{R}$. For each $i \in [m]$, denote $D_i := \{k \in [n] : a_{i,k} \neq b_{i,k}\}$ and d_i to be the largest element in D_i. Then, $|D_i| = \text{dist}(a_i, b_i) = t + s_i$. In addition, we call a pair (I_1, I_2) *compatible with* i if $I_1 \subseteq D_i, |I_1| \geq t + \frac{s_i+1}{2}, I_2 \subseteq [n] \setminus D_i$, or $I_1 \subseteq D_i \setminus \{d_i\}, |I_1| = t + \frac{s_i}{2}, I_2 \subseteq [n] \setminus D_i$ (the latter happens only when s_i is even). Note that $|I_1| \geq t + \frac{s_i}{2}$ in both cases and $|I_1| = t + \frac{s_i}{2}$ only if $I_1 \subseteq D_i \setminus \{d_i\}$. For every $i \in [m]$ and every such pair (I_1, I_2), define a polynomial on $x \in \mathbb{R}^n$ by

$$f_{i,I_1,I_2}(x) := \prod_{k \in I_1 \cup I_2} (x_k - a_{i,k}) \prod_{k \in D_i \setminus I_1} (x_k - b_{i,k}).$$

Recall Eq. (1). The number of pairs compatible with i is $V_{t+s_i,s_i} 2^{n-(t+s_i)}$. Thus, it suffices to show that all such f_{i,I_1,I_2}s are linearly independent. Indeed, since every f_{i,I_1,I_2} is a multilinear polynomial on n variables, the linear independence implies $\sum_{i=1}^{m} V_{t+s_i,s_i} 2^{n-(t+s_i)} \leq 2^n$, giving Eq. (2).

To show the linear independence, we define, for each $i \in [m]$ and each (I_1, I_2) compatible with i, an $x = x_{i,I_1,I_2} \in \mathbb{R}^n$ by $x_k = a_{i,k}$ for all $k \in D_i \setminus I_1$; $x_k = b_{i,k}$ for all $k \in I_1 \cup ([n] \setminus (D_i \cup I_2))$; $x_k \in X \setminus \{b_{i,k}\}$ arbitrary for all $k \in I_2$. We also need the following ordering of the subsets of $[n]$: for distinct subsets $E, F \subseteq [n]$, we denote $E \prec F$ if $|E| < |F|$ or $|E| = |F|$ and $\max(E \setminus F) > \max(F \setminus E)$. We also write $E \preceq F$ if $E \prec F$ of $E = F$. It is easy to check that $\preceq$ induces a total order of all the subsets of $[n]$. Now, we state the crucial claim for the evaluations of f_{i,I_1,I_2}s (on x_{i,I_1,I_2}s).

Claim 2.4. *Let $i, j \in [m]$ and (I_1, I_2) be compatible with i and (J_1, J_2) be compatible with j. Then,*

(i) for $i = j$, we have $f_{j,J_1,J_2}(x_{i,I_1,I_2}) \neq 0$ if and only if $I_1 = J_1$ and $J_2 \subseteq I_2$;

(ii) for $i \neq j$, we have $f_{j,J_1,J_2}(x_{i,I_1,I_2}) \neq 0$ implies $(D_j \setminus J_1) \cup J_2 \prec (D_i \setminus I_1) \cup I_2$.

Proof. Write $x = x_{i,I_1,I_2}$ for simplicity. First, when $i = j$, since $a_{i,k} = b_{i,k}$ for all $k \in J_2 \subseteq [n] \setminus D_i$,

$$f_{j,J_1,J_2}(x_{i,I_1,I_2}) = f_{i,J_1,J_2}(x) = \prod_{k \in J_1 \cup J_2} (x_k - a_{i,k}) \prod_{k \in D_i \setminus J_1} (x_k - b_{i,k}) = \prod_{k \in J_1} (x_k - a_{i,k}) \prod_{k \in (D_i \setminus J_1) \cup J_2} (x_k - b_{i,k}).$$

This means that $f_{j,J_1,J_2}(x) \neq 0$ if and only if $x_k \neq a_{i,k}$ for all $k \in J_1$ and $x_k \neq b_{i,k}$ for all $k \in (D_i \setminus J_1) \cup J_2$. By the definition of $x = x_{i,I_1,I_2}$, we know that $x_k = a_{i,k}$ for all $k \in D_i \setminus I_1$ and $x_k = b_{i,k}$ for $k \in I_1 \cup ([n] \setminus (D_i \cup I_2))$. So, if $f_{j,J_1,J_2}(x) \neq 0$, then $(D_i \setminus I_1) \cap J_1 = \emptyset$, $I_1 \cap (D_i \setminus J_1) = \emptyset$ and $([n] \setminus (D_i \cup I_2)) \cap J_2 = \emptyset$, i.e. $I_1 = J_1$ and $J_2 \subseteq I_2$. On the other hand, if $I_1 = J_1$ and $J_2 \subseteq I_2$, using that $x_k \neq b_{i,k}$ for all $k \in I_2$, it is easy to see that $f_{j,J_1,J_2}(x) \neq 0$. This demonstrates (i).

For (ii), suppose $f_{j,J_1,J_2}(x) \neq 0$. The goal is to show $(D_j \setminus J_1) \cup J_2 \prec (D_i \setminus I_1) \cup I_2$. The fact that

$$f_{j,J_1,J_2}(x) = \prod_{k \in J_1 \cup J_2} (x_k - a_{j,k}) \prod_{k \in D_j \setminus J_1} (x_k - b_{j,k}) \neq 0$$

implies $x_k \neq a_{j,k}$ for all $k \in J_1 \cup J_2$. In particular, since $x_k = b_{i,k}$ for all $k \in I := I_1 \cup ([n] \setminus (D_i \cup I_2))$ (by the definition of $x = x_{i,I_1,I_2}$), it holds that $b_{i,k} \neq a_{j,k}$ for all $k \in (J_1 \cup J_2) \cap I$, meaning $\mathrm{dist}(b_i, a_j) \geq |(J_1 \cup J_2) \cap I|$. Then, the assumption $\mathrm{dist}(b_i, a_j) \leq t$ implies $|(J_1 \cup J_2) \cap I| \leq t$. Observe that $[n] \setminus I = (D_i \setminus I_1) \cup I_2$, and thus

$$|J_1| + |J_2| = |J_1 \cup J_2| = |(J_1 \cup J_2) \cap I| + |(J_1 \cup J_2) \cap ([n] \setminus I)| \leq t + |[n] \setminus I| = t + |(D_i \setminus I_1) \cup I_2| . \quad (3)$$

Namely, $|J_2| \leq t + |(D_i \setminus I_1) \cup I_2| - |J_1|$. Then, using $|D_j| = t + s_j$ and $|J_1| \geq t + s_j/2$, we obtain

$$|(D_j \setminus J_1) \cup J_2| = |D_j| - |J_1| + |J_2| \leq (t + s_j) - 2|J_1| + t + |(D_i \setminus I_1) \cup I_2| \leq |(D_i \setminus I_1) \cup I_2| . \quad (4)$$

If $|(D_j \setminus J_1) \cup J_2| < |(D_i \setminus I_1) \cup I_2|$, then $(D_j \setminus J_1) \cup J_2 \prec (D_i \setminus I_1) \cup I_2$, and we are done.

From now on, let us assume that $|(D_j \setminus J_1) \cup J_2| = |(D_i \setminus I_1) \cup I_2|$. For simplicity, write $E := (D_j \setminus J_1) \cup J_2$ and $F := (D_i \setminus I_1) \cup I_2$. As $|E| = |F|$, our goal is to show that $E \neq F$ and $\max(E \setminus F) > \max(F \setminus E)$. Note that the derivation of Eq. (4) demonstrates that $|E| = |F|$ only if Eq. (3) is an equality and that $|J_1| = t + s_j/2$. In particular, the former implies that $|(J_1 \cup J_2) \cap ([n] \setminus I)| = |[n] \setminus I|$, i.e. $F = (D_i \setminus I_1) \cup I_2 = [n] \setminus I \subseteq J_1 \cup J_2$; the latter implies $J_1 \subseteq D_j \setminus \{d_j\}$. Hence, $d_j \notin F$ while $d_j \in (D_j \setminus J_1) \cup J_2 = E$, showing $E \neq F$. In addition, $\max(E \setminus F) \geq d_j$ as $d_j \in E \setminus F$. Suppose for contradiction that $\max(F \setminus E) > \max(E \setminus F)$ $(\geq d_j)$. However, since $F \subseteq J_1 \cup J_2 \subseteq (D_j \setminus \{d_j\}) \cup J_2$ and $d_j = \max(D_j)$, we have $\max(F \setminus E) \in J_2 \subseteq E$. This is impossible, so $\max(E \setminus F) > \max(F \setminus E)$ must hold. This shows $(D_j \setminus J_1) \cup J_2 = E \prec F = (D_i \setminus I_1) \cup I_2$, as desired. ■

We now complete the proof by showing that all the f_{i,I_1,I_2}s constructed for $i \in [m]$ and (I_1, I_2) compatible with i are linearly independent. Suppose that $F := \sum_{(j,J_1,J_2)} c_{j,J_1,J_2} f_{j,J_1,J_2}$ is the zero polynomial, where $c_{j,J_1,J_2} \in \mathbb{R}$ for $j \in [m]$ and (J_1, J_2) compatible with j. It suffices to prove $c_{j,J_1,J_2} = 0$ for all (j, J_1, J_2). If not, we pick a triple (i, I_1, I_2) with $c_{i,I_1,I_2} \neq 0$; if there are multiple such (i, I_1, I_2)s, pick the one that minimizes $(D_i \setminus I_1) \cup I_2$ in the total order $\preceq$; if there is still a tie, then pick any of them. Consider $x := x_{i,I_1,I_2}$, and suppose that $c_{j,J_1,J_2} f_{j,J_1,J_2}(x) \neq 0$ for some (j, J_1, J_2). If $i = j$, then Claim 2.4(i) implies $J_1 = I_1$ and $J_2 \subseteq I_2$. Due to the minimality of $(D_i \setminus I_1) \cup I_2$, it must be that $J_2 = I_2$, meaning $(i, I_1, I_2) = (j, J_1, J_2)$. If $i \neq j$, Claim 2.4(ii) implies that $(D_j \setminus J_1) \cup J_2 \prec (D_i \setminus I_1) \cup I_2$. Again, the minimality of $(D_i \setminus I_1) \cup I_2$ implies $c_{j,J_1,J_2} = 0$. But this is impossible as we assumed $c_{j,J_1,J_2} f_{j,J_1,J_2}(x) \neq 0$. Altogether, $c_{j,J_1,J_2} f_{j,J_1,J_2}(x) \neq 0$ implies that $(i, I_1, I_2) = (j, J_1, J_2)$. In addition, Claim 2.4(i) asserts $f_{i,I_1,I_2}(x) \neq 0$. So, $0 = F(x) = \sum_{(j,J_1,J_2)} c_{j,J_1,J_2} f_{j,J_1,J_2}(x) = c_{i,I_1,I_2} f_{i,I_1,I_2}(x)$, and thus $c_{i,I_1,I_2} = 0$. This contradicts our assumption that $c_{i,I_1,I_2} \neq 0$. Therefore, $c_{i,I_1,I_2} = 0$ for all (i, I_1, I_2), and this shows that all the f_{i,I_1,I_2}s are linearly independent. ■

We conclude this section by showing that Theorem 1.1 extends to Hamming balls of radius *at most* t.

Corollary 2.5. *Let $n > t \geq 0$ and X be any set of cardinality $|X| \geq 2$. The Helly number of the family of all Hamming balls of radius at most t in X^n is exactly 2^{t+1}.*

Proof. Let h be this Helly number. It suffices to show $h \leq 2^{t+1}$ as the lower bound follows from Proposition 2.1. We may assume that $0, 1 \in X$. By the definition of h, there exists a minimal family of Hamming balls $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$ in X^n of radius at most t such that every $h-1$ of them intersect while all of them do not. Then, $m \geq h$, and every $m-1$ of these balls intersect (as $\mathcal{B}$ is minimal). For each B_i, let $a_i \in X^n$ be its center and $r_i \in [t]$ be its radius, and $b_i \in \bigcap_{j \neq i} B_j \setminus B_i$. Thus, $\mathrm{dist}(a_i, b_i) > r_i$ for $i \in [m]$ and $\mathrm{dist}(a_i, b_j) \leq r_i$ for $i \neq j$. Now, for $1 \leq i \leq m$, we create $(t - r_i)$ more coordinates to all the $2m$ vectors by appending $(t - r_i)$ 1s to a_i and $(t - r_i)$ 0s to all others. In the end, we obtain $a'_1, a'_2, \ldots, a'_m, b'_1, b'_2, \ldots, b'_m \in X^{n'}$ where $n' = n + \sum_{i=1}^m (t - r_i)$ such that $\mathrm{dist}(a'_i, b'_j) = \mathrm{dist}(a_i, b_j) + t - r_i$ for all $1 \leq i, j \leq m$. This means $\mathrm{dist}(a'_i, b'_i) > t$ for $i \in [m]$ and $\mathrm{dist}(a'_i, b'_j) \leq t$ for $i \neq j$. So, $h \leq m \leq 2^{t+1}$ by Theorem 2.3, as desired. ■

3 Binary strings

This section deals with the binary setting, e.g. $X = \{0, 1\}$. In this case, we can prove a stronger result (Theorem 1.5) where we only assume that $\mathrm{dist}(a_i, b_j) + \mathrm{dist}(a_j, b_i) \leq 2t$ for $i \neq j$. The lower bound is, again, derived from Proposition 2.1. For the upper bound, we provide a probabilistic proof that is simpler than that of Theorem 2.3.

Theorem 3.1. *Let $n > t \geq 0$. Suppose $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_m \in \{0,1\}^n$ satisfy for all $i \in [m]$, $\mathrm{dist}(a_i, b_i) = t + s_i$ for some $s_i \geq 1$, and $\mathrm{dist}(a_i, b_j) + \mathrm{dist}(a_j, b_i) \leq 2t$ for all distinct $i, j \in [m]$. Then,*

$$\sum_{i=1}^{m} \frac{V_{t+s_i, s_i}}{2^{t+s_i}} \leq 1. \tag{5}$$

In particular, $f'(t; \{0,1\}) \leq 2^{t+1}$ and $f'(t, s; X) \leq 2^{t+s}/V_{t+s,s}$ if $s_i \geq s$ for all $i \in [m]$.

Proof. Given Eq. (5), the derivation of the bounds for $f'(t; \{0,1\})$ and $f'(t, s; \{0,1\})$ is the same as that in the proof of Theorem 2.3, so we omit it here.

We have to prove Eq. (5). For each i, denote $D_i := \{k \in [n] : a_{i,k} \neq b_{i,k}\}$ and $d_i := \max(D_i)$. Then, $|D_i| = \mathrm{dist}(a_i, b_i) = t + s_i$. Now, sample a string α, uniformly in $\{0,1\}^n$. For each $i \in [m]$, let $D_i(\alpha) := \{k \in D_i : \alpha_k = a_{i,k}\}$. Denote $\mathcal{E}_i$ to be the event that either $|D_i(\alpha)| \geq t + \frac{s_i+1}{2}$, or $|D_i(\alpha)| = t + \frac{s_i}{2}$ and $d_i \notin D_i(\alpha)$ (the latter happens only when s_i is even). In both cases, $|D_i(\alpha)| \geq t + \frac{s_i}{2}$. It suffices to establish Claim 3.2 below because then, $1 \geq \Pr[\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m] = \sum_{i=1}^m \Pr[\mathcal{E}_i] = \sum_{i=1}^n \frac{V_{t+s_i, s_i}}{2^{t+s_i}}$, using Eq. (1).

Claim 3.2. *The events $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_m$ are pairwise disjoint.*

Proof. Suppose for contradiction that $\mathcal{E}_i$ and $\mathcal{E}_j$ are not disjoint for some $1 \leq i < j \leq m$. Let $\alpha \in \mathcal{E}_i \cap \mathcal{E}_j$. For convenience, denote $D_{ij} := D_i \setminus D_j$ and $D_{ji} := D_j \setminus D_i$. By the definition of D_i, we know that a_i and b_i are the same restricted to D_i^c while they are the opposite restricted to D_i, i.e. $a_i\big|_{D_i^c} = b_i\big|_{D_i^c}$ and $a_i\big|_{D_i} = \overline{b_i}\big|_{D_i}$ (we use $\overline{\cdot}$ for the opposite string of the same index set). Similarly, $a_j\big|_{D_j^c} = b_j\big|_{D_j^c}$ and $a_j\big|_{D_j} = \overline{b_j}\big|_{D_j}$. As a consequence, coordinates among $D_{ij} \cup D_{ji}$ will contribute to the distance $\mathrm{dist}(a_i, b_j) + \mathrm{dist}(b_i, a_j)$. More precisely,

$$\mathrm{dist}(a_i\big|_{D_{ij}}, b_j\big|_{D_{ij}}) + \mathrm{dist}(b_i\big|_{D_{ij}}, a_j\big|_{D_{ij}}) = \mathrm{dist}(\overline{b_i}\big|_{D_{ij}}, b_j\big|_{D_{ij}}) + \mathrm{dist}(b_i\big|_{D_{ij}}, b_j\big|_{D_{ij}}) = |D_{ij}|, \tag{6}$$

and similarly,

$$\mathrm{dist}(a_i\big|_{D_{ji}}, b_j\big|_{D_{ji}}) + \mathrm{dist}(b_i\big|_{D_{ji}}, a_j\big|_{D_{ji}}) = |D_{ji}|. \tag{7}$$

Write $g := |D_i \cap D_j|$, and hence $|D_{ij}| = t + s_i - g, |D_{ji}| = t + s_j - g$. Then, Eqs. (6) and (7) imply

$$\begin{aligned}
&\mathrm{dist}(a_i, b_j) + \mathrm{dist}(b_i, a_j) \\
&\geq \mathrm{dist}(a_i\big|_{D_{ij}\cup D_{ji}}, b_j\big|_{D_{ij}\cup D_{ji}}) + \mathrm{dist}(b_i\big|_{D_{ij}\cup D_{ji}}, a_j\big|_{D_{ij}\cup D_{ji}}) \\
&= \mathrm{dist}(a_i\big|_{D_{ij}}, b_j\big|_{D_{ij}}) + \mathrm{dist}(b_i\big|_{D_{ij}}, a_j\big|_{D_{ij}}) + \mathrm{dist}(a_i\big|_{D_{ji}}, b_j\big|_{D_{ji}}) + \mathrm{dist}(b_i\big|_{D_{ji}}, a_j\big|_{D_{ji}}) \\
&= |D_{ij}| + |D_{ji}| = 2t - 2g + s_i + s_j.
\end{aligned} \tag{8}$$

By assumption, $2t \geq 2t - 2g + s_i + s_j$, i.e. $2g \geq s_i + s_j$.

The second step is to consider the contribution to $\mathrm{dist}(a_i, b_j) + \mathrm{dist}(b_i, a_j)$ from $k \in D_i \cap D_j$. Denote $D := D_i(\alpha) \cap D_j(\alpha) \subseteq D_i \cap D_j$. Observe that $a_{i,k} = a_{j,k} = \alpha_k \neq b_{i,k} = b_{j,k}$ for $k \in D$. So

$$\mathrm{dist}(a_i\big|_D, b_j\big|_D) + \mathrm{dist}(b_i\big|_D, a_j\big|_D) = 2|D|. \tag{9}$$

We then lower bound $|D|$. Since $\alpha \in \mathcal{E}_i \cap \mathcal{E}_j$, $|D_i(\alpha)| \geq t + \frac{s_i}{2}$ and $|D_j(\alpha)| \geq t + \frac{s_j}{2}$. Writing $D'_i := D_i(\alpha) \cap (D_i \cap D_j)$ and $D'_j := D_j(\alpha) \cap (D_i \cap D_j)$, we know

$$\begin{cases} |D'_i| = |D_i(\alpha) \cap (D_i \cap D_j)| = |D_i(\alpha) \setminus D_{ij}| \geq t + \dfrac{s_i}{2} - (t + s_i - g) = g - \dfrac{s_i}{2} \\ |D'_j| = |D_j(\alpha) \cap (D_i \cap D_j)| = |D_j(\alpha) \setminus D_{ji}| \geq t + \dfrac{s_j}{2} - (t + s_j - g) = g - \dfrac{s_j}{2}, \end{cases} \tag{10}$$

and thus

$$\begin{aligned}
|D| &= |D'_i \cap D'_j| = |D'_i| + |D'_j| - |D'_i \cup D'_j| \\
&\geq |D'_i| + |D'_j| - |D_i \cap D_j| \geq g - \frac{s_i}{2} + g - \frac{s_j}{2} - g = g - \frac{s_i + s_j}{2}.
\end{aligned} \tag{11}$$

We note that the RHS of Eq. (11) is non-negative because $2g \geq s_i + s_j$. Using Eqs. (6) to (9) and (11),

$$\begin{aligned}
2t &\geq \mathrm{dist}(a_i, b_j) + \mathrm{dist}(b_i, a_j) \\
&\geq \mathrm{dist}(a_i\big|_{D_{ij}\cup D_{ji}}, b_j\big|_{D_{ij}\cup D_{ji}}) + \mathrm{dist}(b_i\big|_{D_{ij}\cup D_{ji}}, a_j\big|_{D_{ij}\cup D_{ji}}) + \mathrm{dist}(a_i\big|_D, b_j\big|_D) + \mathrm{dist}(b_i\big|_D, a_j\big|_D) \\
&\geq |D_{ij}| + |D_{ji}| + 2|D| = (t + s_i - g) + (t + s_j - g) + 2g - (s_i + s_j) = 2t.
\end{aligned}$$

This being an equality implies, in particular, Eq. (11) is an equality, so Eq. (10) is also an equality. The former means $D'_i \cup D'_j = D_i \cap D_j$ while the latter means $D_{ij} \subseteq D_i(\alpha), D_{ji} \subseteq D_j(\alpha)$ and $|D_i(\alpha)| = t + \frac{s_i}{2}, |D_j(\alpha)| = t + \frac{s_j}{2}$. By the definition of $\mathcal{E}_i$ and $\mathcal{E}_j$, $d_i \notin D_i(\alpha)$ and $d_j \notin D_j(\alpha)$, and thus, $d_i \notin D'_i$ and $d_j \notin D'_j$. Also, as $D_{ij} \subseteq D_i(\alpha)$ and $D_{ji} \subseteq D_j(\alpha)$, it must be that $d_i, d_j \in D_i \cap D_j$. Recall that $d_i = \max(D_i)$ and $d_j = \max(D_j)$. This means $d_i = d_j = \max(D_i \cap D_j)$. However, as discussed before, $d_i = d_j \notin D'_i \cup D'_j$. This contradicts that $D'_i \cup D'_j = D_i \cap D_j$. Therefore, $\mathcal{E}_i$ and $\mathcal{E}_j$ must be disjoint for all $1 \leq i < j \leq m$. ■

■

Next, we discuss the tightness of Theorems 2.3 and 3.1. For $1 \leq d \leq n$, an (n, d) error correcting code (ECC) is a collection of binary strings (codewords) of length n with all pairwise distances at least d. Write $A(n, d)$ for the maximum possible size of such a collection. Taking $n = t + s$, a_i to be any one of the codewords and $b_i = \overline{a_i}$, it is easy to see that $f'(t, s, \{0, 1\}) \geq A(t + s, s)$. As discussed in the Introduction, the upper bound $f'(t, s; \{0, 1\}) \leq \frac{2^{t+s}}{V_{t+s,s}}$ is the Hamming bound for ECC when s is odd. Thus, we can use perfect codes (ECCs that match the Hamming bound) and their extensions (add a parity bit so that the length and the distance increases by one while the number of codewords stays the same) to show our bound on $f'(t, s, \{0, 1\})$ is tight, (and the same also holds for $f(t, s, X)$). More precisely,

- $f'(t, s, \{0,1\} = 2^{t+1}$ for $s \in \{1,2\}$. We can take the trivial ECC, all the binary strings of length $t+1$. There are 2^{t+1} of them and all pairwise distances are at least 1. So, $f'(t, 1, \{0,1\}) = A(t+1, 1) = 2^{t+1}$. Adding a parity bit to all these strings, the pairwise distances are at least 2. So, $f'(t, 2, \{0,1\}) = A(t+2, 2) = 2^{t+1}$.

- $f'(t, s, \{0,1\} = \frac{2^{t+3}}{t+4}$ when $s \in \{3,4\}$ and $t+4$ is a power of 2. When $t+4$ is a power of 2, we take the Hamming code: $\frac{2^{t+3}}{t+4}$ binary strings of length $t+3$ and pairwise distances at least 3. This shows that $f'(t, 3, \{0,1\} = A(t+3, 3) = \frac{2^{t+3}}{t+4}$. Adding a parity bit to all these strings, the pairwise distances are at least 4. So, $f'(t, 4, \{0,1\}) = A(t+4, 4) = \frac{2^{t+3}}{t+4}$.

- $f'(16, 7; \{0,1\}) = f'(16, 8; \{0,1\}) = 2048$. Here, we take the Golay code [Gol49]: 2048 binary strings of length 23 whose pairwise distances are at least 7. This, as wells as its extension, implies $f'(16, 7; \{0,1\}) = f'(16, 8; \{0,1\}) = 2048$.

Besides the perfect codes, we can take the Bose–Chaudhuri–Hocquenghem codes (BCH codes) [Hoc59, BRC60]. These are $\Omega(2^{t+s}/(t+s)^s)$ binary strings of length $t+s$ and pairwise distances at least s whenever s is odd. Based on the former discussion, this, and its extension, demonstrate that for every fixed s, $f(t, s; X) = \Theta_s(2^{t+s}/V_{t+s,s})$ and $f'(t, s; X) = \Theta_s(2^{t+s}/V_{t+s,s})$.

We note that our probabilistic proof relies crucially on the fact that each coordinate is either 0 or 1. A similar proof by sampling $\alpha \in X^n$ appropriately works for general Xs but only gives an upper bound of $|X|^{t+1}$. This is not merely a coincidence: when $|X| \in \{3,4\}$, unlike $f(t; X) = 2^{t+1}$, we can prove that $f'(t; X) = \Theta(3^t)$.

Theorem 3.3. *$3^t \leq f'(t; X) \leq 3^{t+1}$ for every $t \geq 0$ and every X of size 3 or 4.*

Proof. For the lower bound, we assume $\{0,1,2\} \subset X$. Define $\varphi : \{0,1,2\} \to \{0,1,2\}$ by $\varphi(0) = 1, \varphi(1) = 2$ and $\varphi(2) = 0$. Let $a_1, a_2, \ldots, a_m$ be an enumeration of $s \in \{0,1,2\}^{t+1}$ such that $\sum_{k=1}^{t+1} s_k$ is a multiple of 3. Since, for every choice of $s_1, \ldots, s_t$ there is a unique s_{t+1} such that $\sum_{k=1}^{t+1} s_k$ is a multiple of 3, we have that $m = 3^t$. For each $i \in [m]$, define $b_i \in \{0,1,2\}^{t+1}$ by $b_{i,k} = \varphi(a_{i,k})$ for all $k \in [t+1]$. Clearly $\mathrm{dist}(a_i, b_i) = t+1$ for all $i \in [m]$, and for any $i \neq j$, it holds that

$$\begin{aligned}\mathrm{dist}(a_i, b_j) + \mathrm{dist}(a_j, b_i) &= |\{1 \leq k \leq t+1 : a_{i,k} \neq \varphi(a_{j,k})\}| + |\{1 \leq k \leq t+1 : \varphi(a_{i,k}) \neq a_{j,k}\}| \\ &= t+1 + \{k : a_{i,k} = a_{j,k}\}.\end{aligned}$$

Since $\sum_{k=1}^{t+1} a_{i,k}$ and $\sum_{k=1}^{t+1} a_{j,k}$ are both multiples of 3, a_i and a_j can share at most $t+1-2 = t-1$ bits. Therefore, $\mathrm{dist}(a_i, b_j) + \mathrm{dist}(a_j, b_i) \leq t+1+t-1 = 2t$. This shows that $f'(t; X) \geq 3^t$.

We now prove the upper bound. Suppose $n > t$ and $a_1, a_2, \ldots, a_m, b_1, b_2 \ldots, b_m \in X^n$ with $\mathrm{dist}(a_i, b_i) \geq t+1$ for $i \in [m]$ and $\mathrm{dist}(a_i, b_j) + \mathrm{dist}(a_j, b_i) \leq 2t$ for $i \neq j$. For $1 \leq k \leq n$, independently and uniformly sample a 2-element subset X_k of X. For each $i \in [m]$, define $a'_i \in \{0,1\}^n$ by $a'_{i,k} := 1$ iff $a_{i,k} \in X_i$ for all $k \in [n]$, and $b'_i \in \{0,1\}^n$ by $b'_{i,k} := 1$ iff $b_{i,k} \in X_i$ for all $k \in [n]$. Denote $I := \{i \in [m] : \mathrm{dist}(a'_i, b'_i) \geq t+1\}$. Using $|X| \in \{3,4\}$, it holds that $\Pr[a'_{i,k} \neq b'_{i,k}] = 2\binom{|X|-2}{1}/\binom{|X|}{2} = \frac{2}{3}$ whenever $a_{i,k} \neq b_{i,k}$. So, $\Pr[i \in I] \geq \left(\frac{2}{3}\right)^{t+1}$, and hence, $\mathbb{E}|I| \geq m\left(\frac{2}{3}\right)^{t+1}$. Observe also that $\mathrm{dist}(a'_i, b'_j) \leq \mathrm{dist}(a_i, b_j) \leq t$ for any distinct $i, j \in I$, which means $|I| \leq f'(t; \{0,1\}) \leq 2^{t+1}$. Therefore, $m(\frac{2}{3})^{t+1} \leq \mathbb{E}|I| \leq 2^{t+1}$, i.e. $m \leq 3^{t+1}$. ■

We remark that for general X, the same argument (by sampling $X_k \in \binom{X}{\lfloor |X|/2 \rfloor}$ instead) shows $3^t \leq f'(n, t, X) \leq \left(\frac{|X|(|X|-1)}{\lfloor |X|/2 \rfloor \lceil |X|/2 \rceil}\right)^{t+1}$. We do not know which of these bounds is closer to the truth.

3.1 A set-pair result

As mentioned in the Introduction, Füredi [Für84] proved that if $A_1, A_2, \ldots, A_m$ are sets of size a and $B_1, B_2, \ldots, B_m$ are sets of size b such that $|A_i \cap B_i| \leq k$ for $i \in [m]$ and $|A_i \cap B_j| > k$ for distinct $i, j \in [m]$, then $m \leq \binom{a+b-2k}{a-k}$, and this is tight. In the same paper, he raised the question of understanding the largest possible size of a family $(A_i, B_i)_{i=1}^m$ such that $|A_i| = a, |B_i| = b, |A_i \cap B_i| \leq \ell$ for all $i \in [m]$ and $|A_i \cap B_j| > k$, (where $k \geq \ell$ are given) for all distinct $i, j \in [m]$. His result shows that this maximum is exactly $\binom{a+b-2\ell}{a-\ell}$ in case $k = \ell$. For the general case, Zhu [Zhu95] showed the answer is at most $\min(\binom{a+b-2\ell}{a-k}/\binom{a-\ell}{k-\ell}, \binom{a+b-2\ell}{b-k}/\binom{b-\ell}{k-\ell})$, and this is tight if there is a collection $\mathcal{A}$ of subsets of $U := [a+b-2\ell]$, each with size $a-\ell$, such that every subset of U with size $a-k$ is contained in exactly one of $\mathcal{A}$, or there is a collection $\mathcal{B}$ of subsets of U with size $b-\ell$, such that every subset of U with size $b-k$ is contained in exactly one of $\mathcal{B}$. These collections are called *designs* or *Steiner systems* and exist when $a-\ell$ is sufficiently larger than $b-\ell$ or $b-\ell$ is sufficiently larger than $a-\ell$ provided the appropriate divisibility conditions hold; see [Kee14, GKLO23].

We note that with a slight change of his argument, we can show the answer is at most $\frac{\binom{a+b-2\ell}{a-\ell-x+y}}{\binom{a-\ell}{x}\binom{b-\ell}{y}}$ for every $x, y \geq 0$ with $x + y = k - \ell$. This is $\exp(O(k-\ell))$ better than his original bound if, say, $a - \ell = b - \ell \gg k - \ell$. Indeed, by the general position and the dimension reduction arguments, used in [Für84, Zhu95], we can essentially assume $\ell = 0$ (with $a-\ell, b-\ell, k-\ell$ replacing a, b, k). For each $i \in [m]$, we build $\binom{a}{x}\binom{b}{y}$ pairs of sets based on (A_i, B_i) by shifting, in all possible ways, a subset X of x elements of A_i from A_i to B_i, and a subset Y of y elements of B_i from B_i to A_i. This gives $m\binom{a}{x}\binom{b}{y}$ pairs $(A_i^{X,Y}, B_i^{X,Y})$ with $|A_i^{X,Y}| = a - x + y, |B_i^{X,Y}| = b - y + x, A_i^{X,Y} \cap B_i^{X,Y} = \emptyset$. We also have $|A_i^{X,Y} \cap B_i^{X',Y'}| > 0$ if $X \neq X'$ and $|A_i^{X,Y} \cap B_j^{X',Y'}| > 0$ if $i \neq j$, since $|X| \leq k - |Y'| = |X'|$. Now, we can apply the result of Bollobás [Bol65] to conclude that $m\binom{a}{x}\binom{b}{y} \leq \binom{a+b}{a-x+y}$, as desired.

Moreover, Theorem 3.1 gives the following variation of Füredi's question where instead of $|A_i| = a, |B_i| = b$, we only require $|A_i| + |B_i| = s$.

Theorem 3.4. *Let $s > k \geq \ell \geq 0$ and $m \geq 0$. Suppose $A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m$ are sets such that $|A_i| + |B_i| = s$ for all $i \in [m]$, $|A_i \cap B_i| \leq \ell$ for all $i \in [m]$ and $|A_i \cap B_j| + |A_j \cap B_i| \geq 2(k+1)$ for all $1 \leq i < j \leq m$. Then, $m \leq f'(s-2(k+1), 2(k+1)-2\ell; \{0,1\}) \leq \frac{2^{s-2\ell-1}}{\sum_{i=0}^{k-\ell}\binom{s-2\ell-1}{i}}$.*

Proof. Suppose all sets are subsets of $[n]$ for some $n \in \mathbb{N}$. For $1 \leq i \leq m$, let $a_i \in \{0,1\}^n$ be the indicator vector of A_i, i.e. $a_{i,k} = 1$ iff $k \in A_i$; similarly, let $b_i \in \{0,1\}^n$ be that of B_i. Then, for $i \neq j$,

$$\begin{cases} \operatorname{dist}(a_i, b_i) = |A_i| + |B_i| - 2|A_i \cap B_i| \geq s - 2\ell, \\ \operatorname{dist}(a_i, b_j) + \operatorname{dist}(a_j, b_i) = |A_i| + |B_j| - 2|A_i \cap B_j| + |A_j| + |B_i| - 2|A_j \cap B_i| \leq 2s - 4(k+1). \end{cases}$$

So,

$$m \leq f'(s-2(k+1), 2(k+1)-2\ell; \{0,1\}) \leq \frac{2^{s-2\ell}}{V_{s-2\ell,2(k+1)-2\ell}} = \frac{2^{s-2\ell-1}}{V_{s-2\ell-1,2(k+1)-2\ell-1}} = \frac{2^{s-2\ell-1}}{\sum_{i=0}^{k-\ell}\binom{s-2\ell-1}{i}}.$$

■

Note that this bound is close to being tight when $s - 2\ell \gg k - \ell$. In this case, we can take the BCH code of length $s - 2\ell - 1$ and pairwise distances at least $2k - 2\ell + 1$. Appending to each codeword a parity bit, we get $\Omega(2^{s-2\ell}/(s-2\ell)^{k-\ell})$ binary strings of length $s - 2\ell$ and pairwise distances at least $2k - 2\ell + 2$. Now, take $A_i \subseteq [s-2\ell]$ to be the set corresponding to each codeword joined with $\{-1, -2, \ldots, -\ell\}$ and $B_i := ([s-2\ell] \setminus A_i) \cup \{-1, -2, \ldots, -\ell\}$. Then, $|A_i| + |B_i| = s, |A_i \cap B_i| = \ell$ for $i \in [m]$ and $|A_i \cap B_j| + |A_j \cap B_i| \geq 2\ell + 2(k - \ell + 1) = 2k + 2$ for $i \neq j$, forming the desired family.

4 Related questions

4.1 Fractional-Helly-type and Hadwiger-Debrunner-type problems

We say a point *hits* a Hamming ball if the ball contains the point and a set of points *hits* a collection of Hamming balls if every ball contains some point in the set.

Theorem 4.1. *Let $m \geq 1, n > t \geq 0$, X be any nonempty set and $B_1, \ldots, B_m$ be Hamming balls of radius t centered at $a_1, \ldots, a_m \in X^n$, respectively.*

(1) *If X is finite and, for some $\alpha > \frac{12}{m}$, at least $\alpha\binom{m}{2}$ (unordered) pairs of the Hamming balls intersect, then some point in X^n hits an $\Omega\big(\alpha^2|X|^{-t}/\binom{4t}{t}\big)$ fraction of the balls.*

(2) *If, for some $\alpha > 0$, at least $\alpha\binom{m}{t+2}$ unordered $(t+2)$-tuples[2] of the Hamming balls intersect, then some point in X^n hits an $\Omega(\alpha/(e(t+1))^{t+1})$ fraction of the balls.*

Theorem 4.2. *Let $m \geq 1, n > t \geq 0, p \geq q \geq 2$, and X be a nonempty set. Let $B_1, B_2, \ldots, B_m$ be Hamming balls of radius t centered at $a_1, a_2, \ldots, a_m \in X^n$, respectively, where out of any p balls, q of them intersect.*

- *If $|X| < \infty$ and $q \leq t+1$, then there exist $p^q q^t |X|^{t+2-q} 2^{O(t)}$ points in X^n hitting all these Hamming balls;*
- *if $q = t+2$, then there exist $O(e^{2t}p^{t+1})$ points in X^n hitting all these Hamming balls.*

Remark. Suppose $B_1, B_2, \ldots, B_m$ are Hamming balls of radius t. If every $t+2$ of them intersect, then there exist $O_t(1)$ points in X^n hitting all of them (by Theorem 4.2). On the other hand the number $t+2$ cannot be replaced by $t+1$, as can be seen by taking $n = t+1$ and an infinite X. But if any 2^{t+1} of them intersect, then all of them intersect (Theorem 1.1). This behavior differs from the setting of Helly's theorem, where a family of convex sets in $\mathbb{R}^d$ is considered: if every $d+1$ of them intersect, then all intersect, but even if every d of them intersect, there still can be no finite bound for the minimum number of points required to hit all of them.

We first give a simple proof for Theorem 4.1(1). To this end, we need the following lemma whose proof is delayed.

Lemma 4.3. *Let $n > t \geq \delta \geq 0$, X be a finite nonempty set, and $a, b \in X^n$. Then, there is a set of $\binom{4t-\delta}{t-\delta}|X|^{t-\delta}$ points in X^n hitting all the Hamming balls $B(p,t)$ with $dist(a,p) \leq \min(dist(a,b), 2t-\delta)$ and $dist(b,p) \leq 2t$.*

We remark that when $t = \delta$, we do not require $|X| < \infty$ because then, all $B(p,t)$s under consideration contains a.

Proof for Theorem 4.1(1). We may assume $m \geq 12$ as otherwise $\alpha > 1$. Construct a graph G with vertex set $V(G) = [m]$ where i and j are adjacent if B_i and B_j intersect, i.e. $\mathrm{dist}(a_i, a_j) \leq 2t$. Starting from G, by iteratively deleting vertices of degree smaller than $\alpha(m-1)/2$ as long as there is one, we arrive at an induced subgraph G' of G. By assumption, $e(G) \geq \alpha\binom{m}{2} = m \cdot \alpha(m-1)/2$. This means G' is not empty and hence, the minimum degree of G' is at least $\alpha(m-1)/2 \geq \alpha m/3$ (using $m \geq 12$). Fix any vertex $u \in V(G')$. The number of paths uvw in G' is at least $\alpha m/3 \cdot (\alpha m/3 - 1) \geq \alpha^2 m^2/12$, using $\alpha \geq 12/m$.

An (ordered) triple of distinct vertices $(x, y, z) \in V(G')^3$ is said to be *good* if $\mathrm{dist}(a_x, a_z) \leq \min(\mathrm{dist}(a_x, a_y), 2t)$ and $\mathrm{dist}(a_y, a_z) \leq 2t$. Observe that for any path uvw in G,

[2]The $t+2$ entries are distinct.

- if $uw \notin E(G)$, then $\mathrm{dist}(a_v, a_w) > 2t$ and $\mathrm{dist}(a_u, a_v) \le 2t < \mathrm{dist}(a_u, a_w)$, so (u, w, v) is good;
- if $uw \in E(G)$, then $\mathrm{dist}(a_v, a_w) \le 2t, \mathrm{dist}(a_u, a_v) \le \mathrm{dist}(a_u, a_w) \le 2t$ (so (u, w, v) is good) or $\mathrm{dist}(a_v, a_w) \le 2t, \mathrm{dist}(a_u, a_w) \le \mathrm{dist}(a_u, a_v) \le 2t$ (so (u, v, w) is good).

Enumerating over all paths of length 2, there are at least $\alpha^2 m^2/24$ good triples (u, v, w), where u is fixed (as each good triple is counted at most twice). By the pigeonhole principle, there exists $u, v \in [m]$ and $W \subseteq [m]$ such that $|W| \ge \alpha^2 m/24$ and (u, v, w) is good for all $w \in W$. Then, Lemma 4.3 with $\delta = 0, a = a_u, b = a_v, p = a_w$ guarantees $\binom{4t}{t}|X|^t$ points in X^n hitting every B_w, $w \in W$. Therefore, some point among these $\binom{4t}{t}|X|^t$ points hits at least $\frac{\alpha^2 m}{24\binom{4t}{t}|X|^t} = \Omega(\alpha^2 m|X|^{-t}/\binom{4t}{t})$ balls. ■

We now provide the following definitions that are useful in the proof of Theorem 4.1(1) and of Theorem 4.2.

Definition 4.4. *Let $m \ge 0, n > t \ge 0$, let X be a nonempty set, and $a_1, \ldots, a_m \in X^n$. Define $\varphi(a_1, a_2, \ldots, a_m; t)$ to be the largest size of $K \subseteq [n]$ such that for some $w \in X^n$, $\mathrm{dist}(w\big|_{K^c}, a_i\big|_{K^c}) + |K| \le t$ for all $i \in [m]$. Define $\varphi(a_1, a_2, \ldots, a_m; t) := -\infty$ if no such K exists.*

This definition says that, without looking at the coordinates indexed by $k \in K$, there exists some point lying in all the Hamming balls of radius $t - |K|$ centered at a_i, $1 \le i \le m$. In other words, we can freely choose the coordinates in K for w while maintaining that $w \in \bigcap_{i=1}^m B(a_i, t)$. We note that $\varphi(a_1, \ldots, a_{m+1}; t) \le \varphi(a_1, \ldots, a_m; t) \le t$, $\varphi(a_1; t) = t$ and $\varphi(a_1, \ldots, a_m; t) \ge 0$ if and only if $\bigcap_{i=1}^m B(p_i, t) \ne \emptyset$. In addition, we can assume that $w_k \in \{a_{1,k}, a_{2,k}, \ldots, a_{m,k}\}$ for all $k \in [n] \setminus K$ because otherwise, we should have considered $K' := K \cup \{k\}$. This motivates the following definition.

Definition 4.5. *Let $m \ge 0, n > t \ge 0$, let X be a nonempty set, and $a_1, \ldots, a_m \in X^n$. Define $W(a_1, a_2, \ldots, a_m; t)$ to be the set of $w \in \bigcap_{i=1}^m B(a_i, t)$ where $w_k \in \{a_{1,k}, a_{2,k}, \ldots, a_{m,k}\}$ for all $k \in [n]$.*

When it is clear from the context, we omit t in $\varphi(\cdot)$ and $W(\cdot)$. The following crucial property, whose proof is delayed, shows how to use $\varphi(\cdot)$ in order to find a "small" set hitting the Hamming balls.

Lemma 4.6. *Let $m \ge 1, n > t \ge 0$, let X be any nonempty set, and $a_1, \ldots, a_m \in X^n$.*

(1) $|W(a_1, a_2 \ldots, a_m)| \le (em)^t$;

(2) $W(a_1, a_2 \ldots, a_m)$ hits $B(a, t)$ for any $a \in X^n$ with $\varphi(a_1, a_2, \ldots, a_m, a) = \varphi(a_1, a_2, \ldots, a_m) \ge 0$.

Now, we can prove Theorem 4.1(2) and Theorem 4.2.

Proof of Theorem 4.1(2). We may assume $m \ge 2t$ without loss of generality. An unordered tuple $(i_1, i_2, \ldots, i_{t+2}) \in \binom{[m]}{t+2}$ is said to be *good* if $B_{i_1}, B_{i_2}, \ldots, B_{i_{t+2}}$ intersect. Find the largest $\ell \in [t+1]$ such that there exists (distinct) $i_1, i_2 \ldots, i_\ell \in [m]$ with the following properties.

- $0 \le \varphi(a_{i_1}, a_{i_2}, \ldots, a_{i_j}) < \varphi(a_{i_1}, a_{i_2}, \ldots, a_{i_{j-1}})$ for $2 \le j \le \ell$;
- there are at least $\frac{t+3-\ell}{t+2}\alpha\binom{m-\ell}{t+2-\ell}$ good tuples containing $i_1, \ldots, i_\ell$.

We first note that ℓ is well-defined because when $\ell = 1$, the pigeonhole principle implies that some $i_1 \in [m]$ lies in at least $\alpha\binom{m}{t+2} \cdot \frac{t+2}{m} = \alpha\binom{m-1}{t+1}$ good tuples. Now, let I be the set of $i_{\ell+1} \in [m] \backslash \{i_1, i_2, \ldots, i_\ell\}$

such that at least $\frac{t+2-\ell}{t+2}\alpha\binom{m-\ell-1}{t+1-\ell}$ good tuples contain $i_1, i_2, \ldots, i_{\ell+1}$. Let Z be the count of the number of good tuples containing $i_1, i_2, \ldots, i_\ell$ along with one entry other than $i_1, i_2, \ldots, i_\ell$. It holds that

$$\begin{aligned}\frac{t+3-\ell}{t+2}\alpha\binom{m-\ell}{t+2-\ell}\cdot(t+2-\ell) \le Z &\le |I|\binom{m-\ell-1}{t+1-\ell} + (m-\ell-|I|)\frac{t+2-\ell}{t+2}\alpha\binom{m-\ell-1}{t+1-\ell}\\ &\le |I|\binom{m-\ell-1}{t+1-\ell} + \frac{t+2-\ell}{t+2}\alpha\binom{m-\ell}{t+2-\ell}(t+2-\ell),\end{aligned}$$

implying $|I| \ge \frac{\alpha}{t+2}\binom{m-\ell}{t+2-\ell}(t+2-\ell)/\binom{m-\ell-1}{t+1-\ell} = \frac{\alpha}{t+2}(m-\ell)$. We claim that $\varphi(a_{i_1}, a_{i_2}, \ldots, a_{i_\ell}) = \varphi(a_{i_1}, a_{i_2}, \ldots, a_{i_{\ell+1}})$ for all $i_{\ell+1} \in I$. Indeed, if $\ell = t+1$, then

$$\varphi(a_{i_1}, a_{i_2}, \ldots, a_{i_\ell}) \le \varphi(a_{i_1}, a_{i_2}, \ldots, a_{i_{\ell-1}}) - 1 \le \cdots \le \varphi(a_{i_1}) - (\ell-1) = \varphi(a_{i_1}) - t = 0,$$

so $0 \le \varphi(a_{i_1}, a_{i_2}, \ldots, a_{i_{\ell+1}}) \le \varphi(a_{i_1}, a_{i_2}, \ldots, a_{i_\ell}) = 0$. On the other hand, if $\ell < t+1$, then the claim holds due to the maximality of ℓ (otherwise $i_1, i_2, \ldots, i_{t+1}$ is a longer sequence). Now, Lemma 4.6 guarantees a set of at most $(e\ell)^t$ points in X^n hitting every $B_{i_{\ell+1}}$, $i_{\ell+1} \in I$. By the pigeonhole principle, some point in X^n hits at least $|I|/(e\ell)^t \ge \frac{\alpha(m-\ell)}{(t+2)(e\ell)^t} = \Omega(\alpha m/(e(t+1))^{t+1})$ of the Hamming balls, (here, we used that $m \ge 2t$). ∎

Proof of Theorem 4.2. Write $A := \{a_1, a_2, \ldots, a_m\}$. Recall that given any $x_1, x_2, \ldots, x_k \in A$, $\varphi(x_1, x_2, \ldots, x_k) \ge 0$ if and only if $\bigcap_{i=1}^k B(x_i, t) \ne \emptyset$. Find the largest $\ell \ge 1$ such that there exist $x_1, x_2, \ldots, x_\ell \in A$ with the following properties.

(i) For every $1 \le i_1 < i_2 < \cdots < i_q \le \ell$, it holds that $\bigcap_{j=1}^q B(x_{i_j}, t) = \emptyset$;

(ii) for every $2 \le k < q$ and $1 \le i_1 < i_2 < \cdots < i_k \le \ell$ with $\bigcap_{j=1}^{k-1} B(x_{i_j}, t) \ne \emptyset$, it holds that $\varphi(x_{i_1}, x_{i_2}, \ldots, x_{i_k}) < \varphi(x_{i_1}, x_{i_2}, \ldots, x_{i_{k-1}})$.

We first note that ℓ is well-defined as one can take $\ell = 1, x_1 = a_1$. In addition, by our assumption, out of any p Hammings balls among $B_1, B_2, \ldots, B_m$, q of them intersect, so $\ell \le p-1$. Fix any $a \in A$. The maximality of ℓ implies that $\varphi(x_{i_1}, x_{i_2}, \ldots, x_{i_{q-1}}, a) \ge 0$ for some $1 \le i_1 < i_2 < \cdots < i_{q-1} \le \ell$ or $0 \le \varphi(x_{i_1}, x_{i_2}, \ldots, x_{i_k}, a) = \varphi(x_{i_1}, x_{i_2}, \ldots, x_{i_k})$ for some $1 \le k < q-1$ and $1 \le i_1 < i_2 < \cdots < i_k \le \ell$. As a consequence, one of the following must hold.

(1) $0 \le \varphi(x_{i_1}, x_{i_2}, \ldots, x_{i_k}, a) = \varphi(x_{i_1}, x_{i_2}, \ldots, x_{i_k})$ for some $1 \le k < q, 1 \le i_1 < i_2 < \cdots < i_k \le \ell$;

(2) $0 \le \varphi(x_{i_1}, x_{i_2}, \ldots, x_{i_{q-1}}, a) < \varphi(x_{i_1}, x_{i_2}, \ldots, x_{i_{q-1}})$ for some $1 \le i_1 < i_2 < \cdots < i_{q-1} \le \ell$.

We deal with $a \in A$ satisfying (1) and (2) separately. For every $\emptyset \ne J \subseteq [\ell]$ of size at most $q-1$, let $W_J := W(x_{i_j} : j \in J)$ (see Definition 4.5). By Lemma 4.6, $|W_J| \le (e|J|)^t \le \big((e(q-1)\big)^t$ and W_J hits $B(a,t)$ whenever $a \in A$ fulfills (1) with $J = \{i_1, i_2, \ldots, i_k\}$. Taking the union of all these W_Js, $W := \bigcup_J W_J$ satisfies $|W| \le \binom{p-1}{<q}\big(e(q-1)\big)^t \le \binom{p}{<q}\big(e(q-1)\big)^t$ and W hits $B(a,t)$ whenever $a \in A$ fulfills (1).

Now, suppose $J = \{i_1 < i_2 < \cdots < i_{q-1}\} \subseteq [\ell]$. Consider A_J, the set of all $a \in A$ satisfying (2) with $i_1, i_2, \ldots, i_{q-1}$. We will propose a set $Y_J \subseteq X^n$ that hits every $B(a,t)$, $a \in A_J$. To this end, we may assume $\varphi(x_{i_1}, x_{i_2}, \ldots, x_{i_{q-1}}) \ge 0$ as otherwise $A_J = \emptyset$. We also need the following estimate.

Claim 4.7. *For every $a \in A$, $dist(a, W_J) := \min_{w \in W_J} dist(a, w) \le 2t + 2 - q$.*

Proof. Since $\varphi(x_{i_1}, x_{i_2}, \dots, x_{i_{q-1}}, a) \geq 0$, there exists $w \in B(a,t) \cap \bigcap_{j=1}^{q-1} B(x_{i_j}, t)$, i.e. $\mathrm{dist}(w,a) \leq t$ and $\mathrm{dist}(w, x_{i_j}) \leq t$ for all $j \in [q-1]$. Let K be the set of $k \in [n]$ such that $w_k \notin \{x_{i_1,k}, x_{i_2,k} \dots, x_{i_{q-1},k}\}$. Then, $\mathrm{dist}(w\big|_{K^c}, x_{i_j}\big|_{K^c}) + |K| = \mathrm{dist}(w, x_{i_j}) \leq t$ for all $j \in [q-1]$. Using (ii) and Definition 4.4, we acquire

$$|K| \leq \varphi(x_{i_1}, x_{i_2}, \dots, x_{i_{q-1}}) \leq \varphi(x_{i_1}, x_{i_2}, \dots, x_{i_{q-2}}) - 1 \leq \dots \leq t + 2 - q.$$

Now, pick $w' \in X^n$ where $w'_k = w_k$ for $k \in K^c$ and $w'_k = x_{i_1,k}$ for $k \in K$. It satisfies that $w'_k \in \{x_{i_1,k}, x_{i_2,k}, \dots, x_{i_{q-1},k}\}$ for all $k \in [n]$ and $\mathrm{dist}(w', x_{i_j}) \leq \mathrm{dist}(w'\big|_{K^c}, x_{i_j}\big|_{K^c}) + |K| = \mathrm{dist}(w\big|_{K^c}, x_{i_j}\big|_{K^c}) + |K| = \mathrm{dist}(w, x_{i_j}) \leq t$ for all $j \in [q-1]$. In other words, $w' \in \bigcap_{j=1}^{q-1} B(x_{i_j}, t)$ and hence, $w' \in W_J$. So, $\mathrm{dist}(a, W_J) \leq \mathrm{dist}(a, w') \leq \mathrm{dist}(a, w) + \mathrm{dist}(w, w') \leq t + |K| \leq 2t + 2 - q$. ■

Next, we generate $x_{J,1}, x_{J,2}, \dots, x_{J,k_J} \in A_J$ as follows. Pick any $x_{J,1} \in A_J$; having picked $x_{J,1}, x_{J,2}, \dots, x_{J,k}$ for some $k \geq 1$, if there exists $a \in A_J$ with $\mathrm{dist}(a, x_{J,i}) > 2t$ for all $i \in [k]$, pick $x_{J,k+1}$ to be such a that maximizes $\mathrm{dist}(a, W_J)$. Clearly, among $B(x_{J,1}, t), B(x_{J,2}, t), \dots, B(x_{J,k_J}, t)$, no two balls intersect, so $k_J < p$. For every $w \in W_J$ and every $1 \leq k \leq k_J$, let $Y_{w,k}$ be the set of points given by Lemma 4.3 (plugging $a := w$, $b := x_{J,k}$ and $\delta = q-2$); so $|Y_{w,k}| \leq \binom{4t+2-q}{t+2-q}|X|^{t+2-q}$. We claim that $Y_J := \bigcup_{w \in W_J, k \in [k_J]} Y_{w,k}$ hits every $B(a,t)$, $a \in A_J$. To this end, fix an arbitrary $a \in A_J$. According to the generation of $x_{J,1}, x_{J,2}, \dots, x_{J,k_J}$, there exists $k_a \in [k_J]$ such that $\mathrm{dist}(a, x_{J,k_a}) \leq 2t$. We may take the minimum such k_a. Thus,$\mathrm{dist}(a, x_{J,k}) > 2t$ for all $1 \leq k < k_a$. But then, the procedure (in step k_a) also implies $\mathrm{dist}(a, W_J) \leq \mathrm{dist}(x_{J,k_a}, W_J)$. Taking $w \in W_J$ such that $\mathrm{dist}(a, W_J) = \mathrm{dist}(a, w)$, we acquire $\mathrm{dist}(a, w) = \mathrm{dist}(a, W_J) \leq \mathrm{dist}(x_{J,k_a}, W_J) \leq \mathrm{dist}(x_{J,k_a}, w)$. Recall from Claim 4.7 that $\mathrm{dist}(a, w) = \mathrm{dist}(a, W_J) \leq 2t + 2 - q$. Altogether, $\mathrm{dist}(w, a) \leq \min(\mathrm{dist}(w, x_{J,k_a}), 2t + 2 - q)$ and $\mathrm{dist}(a, x_{J,k_a}) \leq 2t$. By Lemma 4.3, $Y_{w,t}$ hits $B(a,t)$ and hence, Y_J hits $B(a,t)$. In addition,

$$|Y_J| \leq \sum_{w \in W_J, k \in [k_J]} |Y_{w,k}| < |W_J| k_J \binom{4t+2-q}{t+2-q} |X|^{t+2-q} \leq (eq)^t p 2^{O(t)} |X|^{t+2-q} \leq q^t p 2^{O(t)} |X|^{t+2-q}.$$

To complete the proof, we consider two cases. If $q = t+2$, note that all $a \in A$ satisfy (1). Indeed, $a \in A$ satisfying (2) is not possible, since then (2) and (ii) imply

$$\begin{aligned} 0 &\leq \varphi(x_{i_1}, x_{i_2}, \dots, x_{i_{q-1}}, a) \leq \varphi(x_{i_1}, x_{i_2}, \dots, x_{i_{q-1}}) - 1 \leq \varphi(x_{i_1}, x_{i_2}, \dots, x_{i_{q-2}}) - 2 \leq \cdots \\ &\leq \varphi(x_{i_1}) - (q-1) = t - (q-1) = -1. \end{aligned}$$

In other words, $a_1, a_2, \dots, a_m$ all satisfy (1). By the former discussion, W hits every B_i, $i \in [m]$, where

$$|W| \leq \binom{p}{\leq t+1}(e(t+1))^t = O\left(\left(\frac{ep}{(t+1)}\right)^{t+1}(e(t+1))^t\right) = O(e^{2t}p^{t+1}).$$

If $2 \leq q \leq t+1$. Either $a \in A$ satisfies (1), so W hits $B(a,t)$, or $a \in A$ satisfies (2), so Y_J hits $B(a,t)$ for some $J \subseteq [\ell]$ of size $q-1$. Thus, $Y := W \cup \bigcup_J Y_J$ is the desired set, whose size

$$|Y| \leq \binom{p}{< q}\big(e(q-1)\big)^t + \binom{p}{q-1} q^t p 2^{O(t)} |X|^{t+2-q} = p^q q^t |X|^{t+2-q} 2^{O(t)}.$$

■

Proof of Lemma 4.3. Write $P := \{p \in X^n : \mathrm{dist}(a,p) \leq \min(\mathrm{dist}(a,b), 2t - \delta), \mathrm{dist}(b,p) \leq 2t\}$. We may assume that $P \neq \emptyset$ and that $\mathrm{dist}(a,b) > t$ as otherwise every $B(p,t)$, $p \in P$, contains a. By taking any $p \in P$, we know $\mathrm{dist}(a,b) \leq \mathrm{dist}(a,p) + \mathrm{dist}(p,b) \leq 4t - \delta$. Write $D := \{k \in [n] : a_k \neq b_k\}$, so $|D| = \mathrm{dist}(a,b) \le 4t-\delta$. Let Y be the set of all $y \in X^n$ such that $\{k : y_k \ne a_k\}$ is a subset of D of size at most $t-\delta$; $|Y| \le \binom{4t-\delta}{t-\delta}|X|^{t-\delta}$. We prove that Y has the desired property. To this end, fix any $p \in P$, and we will find some $y \in Y$ hitting $B(p,t)$. Writing $D_p := \{k \in [n] : a_k \ne p_k\}$, it holds that $|D_p| = \mathrm{dist}(a,p) \le \min(|D|, 2t-\delta) \le |D|$, thereby $|D \triangle D_p| \ge 2|D_p \setminus D|$. Now, observe that $b_k \ne p_k$ for all $k \in D\triangle D_p$. We then acquire $2|D_p \setminus D| \le |D\triangle D_p| \le \mathrm{dist}(b,p) \le 2t$, i.e. $|D_p \setminus D| \le t$. Now, take any $I \subseteq D_p \cap D$ of size $\min(|D_p \cap D|, t-\delta))$ and $y \in X^n$ such that $y_k = a_k$ for all $k \in [n]\setminus I$ and $y_k = p_k$ for all $k \in I$. We know $y \in Y$ (because $|I| \le t-\delta$) and $\mathrm{dist}(y,p) = |D_p \setminus I| \le t$. Indeed, if $|I| = t-\delta$ it holds because $|D_p| \le 2t-\delta$, otherwise $|I| = |D_p \cap D|$ and hence $|D_p \setminus I| = |D_p \setminus D| \le t$. In other words, $y \in Y$ hits $B(p,t)$, completing the proof. ■

Proof of Lemma 4.6. Write $W := W(a_1, a_2, \dots, a_m)$. For (1), we may assume $W \ne \emptyset$ and $m \ge 2$ because $W = \{a_1\}$ when $m = 1$. For each $k \in [n]$, denote $V_k := \{a_{1,k}, a_{2,k}, \dots, a_{m,k}\}$. Consider $\sum_{i=1}^m \mathrm{dist}(a_i, \tilde{w})$ for an arbitrary $\tilde{w} \in W$, which counts pairs $(i,k) \in [m]\times[n]$ where $a_{i,k} \ne \tilde{w}_k$. For each $k \in [n]$, there are at least $|V_k|-1$ indices $i \in [m]$ with $a_{i,k} \ne w_k$, so $\sum_{k=1}^n (|V_k|-1) \le \sum_{i=1}^m \mathrm{dist}(a_i, \tilde{w}) \le mt$. Now, observe that any $w \in W$ has that $w_k \ne a_{1,k}$ for at most t of $k \in [n]$. Thus, we can enumerate over set of these indices k, which we denote by $S \subseteq [n]$, and for each such $k \in S$, there are $|V_k|-1$ choices for w_k, i.e.

$$|W| \le \sum_{S\subseteq[n],|S|\le t} \prod_{k\in S}(|V_k|-1) \le \sum_{s=0}^{t} \frac{1}{s!}\left(\sum_{k=1}^{n}(|V_k|-1)\right)^s \le \sum_{s=0}^{t}\frac{(mt)^s}{s!} \le 2\frac{(mt)^t}{t!} \le (em)^t.$$

Here, we used the Stirling's approximation $t! \ge 2(t/e)^t$ for $t \ge 1$.

For (2), let $K \subseteq [n], w \in X^n$ be the set and vector in the definition of $\varphi(a_1, a_2, \dots, a_m, a)$. Putting $w_k = a_{1,k}$ for all $k \in K$, we have $\mathrm{dist}(a,w) \le t$ and $\mathrm{dist}(a_i, w) \le t$ for all $i \in [m]$. It suffices to show that $w \in W$. Suppose not, i.e. $w_k \notin \{a_{1,k}, a_{2,k}, \dots, a_{m,k}\}$ for some $k \in [n]$. Clearly, $k \notin K$. Putting $L := K \cup \{k\}$, it holds that $\mathrm{dist}(w\big|_{L^c}, a_i\big|_{L^c}) + |L| = \mathrm{dist}(w\big|_{K^c}, a_i\big|_{K^c}) - 1 + |K| + 1 \le t$ for all $1 \le i \le m$. This means $\varphi(a_1, a_2, \dots, a_m) \ge |L| > \varphi(a_1, a_2, \dots, a_m, a)$, contradicting our assumption. Thus, $w \in W \cap B(a,t)$, as desired. ■

4.2 Sequences of sets

One way to generalize Theorem 1.4 is to consider sequences of sets. More precisely, given $n > t \ge 0$, $a, b \ge 1$ and a set X with $|X| \ge a+b$, an (n,t,a,b,X)-*system* is a collection of pairs $(A_i, B_i)_{i\in[m]}$ (for some m) such that for each i, $A_i = (A_{i,1}, A_{i_2}, \dots, A_{i,n})$ (similarly, $B_i = (B_{i,1}, B_{i_2}, \dots, B_{i,n})$) where each $A_{i,k}$ is a subset of X of size at most a (similarly, each $B_{i,k}$ is a subset of X of size at most b). Define the distance $\mathrm{dist}(A_i, B_j)$ to be the number of $k \in [n]$ such that $A_{i,k} \cap B_{j,k} = \emptyset$. Then, we can extend $f(t;X)$ by denoting $f(n,t,a,b;X)$ to be the size of the largest (n,t,a,b,X)-system such that $\mathrm{dist}(A_i, B_j) \ge t+1$ if and only if $i = j$. One can check that Theorem 2.3 corresponds to the case $a = b = 1$ by replacing each entry of a_is and b_is by a singleton containing it. The first author [Alo85] proved that $f(t+1,t,a,b;X) = \binom{a+b}{a}^{t+1}$ and Theorem 1.4 implies $f(n,t,1,1;X) = 2^{t+1}$. One natural guess might be that $f(n,t,a,b;X) = \binom{a+b}{a}^{t+1}$ for all $n > t \ge 0$. In particular, this would mean that $f(n,t,a,b;X)$ is independent of n. However, this is not the case whenever $a > 1$ or $b > 1$.

Proposition 4.8. $\binom{n}{t+1}\left(\binom{a+b}{b} - 2\right)^{t+1} \le f(n,t,a,b;X) \le \binom{n}{t+1}\binom{a+b}{b}^{t+1}$ *if* $n > t \ge 0$ *and* $|X| \ge a+b$.

Proof. For the upper bound, suppose $(A_i, B_i)_{i\in[m]}$ is an (n,t,a,b,X)-system realizing $f(n,t,a,b;X)$. Uniformly sample a subset $S \subseteq [n]$ of size $t+1$ and consider the following $(t+1,t,a,b,X)$-system:

for each $i \in [m]$, $A'_i := (A_{i,k})_{k\in S}$ and $B'_i := (B_{i,k})_{k\in S}$. Clearly, $\mathrm{dist}(A'_i, B'_j) \le \mathrm{dist}(A_i, B_j) \le t$ for every distinct $i, j \in I$. Let I be the set of $i \in [m]$ where $\mathrm{dist}(A'_i, B'_i) \ge t+1$. We know $|I| \le f(t+1, t, a, b; X) = \binom{a+b}{a}^{t+1}$. Also, using that $\Pr[i \in I] \ge 1/\binom{n}{t+1}$, we conclude that $m/\binom{n}{t+1} \le \mathbb{E}|I| \le \binom{a+b}{a}^{t+1}$, i.e. $f(n, t, a, b; X) = m \le \binom{n}{t+1}\binom{a+b}{b}^{t+1}$.

For the lower bound, we may assume $X = [a+b]$. Let $S_1, \ldots, S_\ell$ be an arbitrary enumeration of all subsets of X of size a (so $\ell = \binom{a+b}{a}$), and for each $i \in [\ell]$, let $T_i := X \setminus S_i$. Observe that $S_i \cap T_j = \emptyset$ if and only if $i = j$. Define a mapping $\varphi : [\ell - 1] \to [\ell]$ by putting $\varphi(\ell - 1) = \ell$ and $\varphi(i) = i$ for all $i \in [\ell - 2]$. Now, let $a_1, \ldots, a_m$ be any enumeration of the sequences in $[\ell-1]^n$, exactly $n-t-1$ entries of which are $\ell - 1$. Then, $m = \binom{n}{t+1}(\binom{a+b}{a} - 2)^{t+1}$. For each $i \in [m]$, define $A_i = (A_{i,k})_{k\in[n]}$ where $A_{i,k} = S_{a_{i,k}}$ and $B_i = (B_{i,k})_{k\in[n]}$ where $B_{i,k} = T_{\varphi(a_{i,k})}$. Since $(A_i, B_i)_{i\in[m]}$ is a (n, t, a, b, X)-system, it suffices to check that $\mathrm{dist}(A_i, B_j) \ge t+1$ if and only if $i = j$. For any $i, j \in [m]$ and $k \in [n]$, it holds that $A_{i,k} \cap B_{j,k} = \emptyset \Leftrightarrow S_{a_{i,k}} \cap T_{\varphi(a_{j,k})} = \emptyset \Leftrightarrow a_{i,k} = a_{j,k} \in [\ell - 2]$. Thus,

$$\mathrm{dist}(A_i, B_j) = |\{k : A_{i,k} \cap B_{j,k} = \emptyset\}| = |\{k : a_{i,k} = a_{j,k} \in [\ell-2]\}| \le |\{k : a_{i,k} \in [\ell-2]\}| = t+1.$$

Clearly, if $i = j$, then $\mathrm{dist}(\mathcal{A}_i, \mathcal{B}_i) = t+1$. On the other hand, if $\mathrm{dist}(\mathcal{A}_i, \mathcal{B}_j) \ge t+1$, it must be that $a_{j,k} = a_{i,k}$ for all k with $a_{i,k} \in [\ell - 2]$. This shows $i = j$. ■

Notably, when $b = 1$ and $|X| = a+1$, $f(n, t, a, 1; X)$ is equal to the maximum m such that there exist $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_m \in X^n$ where $\mathrm{dist}(a_i, b_i) \le n-t-1$ for all i and $\mathrm{dist}(a_i, b_j) \ge n-t$ for all $i \ne j$. Indeed, given $A_{i,k}$ with $|A_{i,k}| = a$ and $B_{i,k}$ with $|B_{i,k}| = 1$, for each $1 \le i \le m$, we can define $a_i = (a_{i,k})_k$ where $a_{i,k}$ is the only element in $X \setminus A_{i,k}$, and $b_i = (b_{i,k})_k$ where $b_{i,k}$ is the only element in $B_{i,k}$. Then, $\mathrm{dist}(A_i, B_j) = n - \mathrm{dist}(a_i, b_j)$. When $a = 1$, the answer is the same as $f(n, t; \{0, 1\}) = 2^{t+1}$ by flipping all the b_is. But when $a > 1$, Proposition 4.8 shows that the largest such family has size $\binom{n}{t+1}(a - O(1))^{t+1}$.

4.3 Connection to the Prague dimension

Given a graph G, the *Prague dimension*, $\mathrm{pd}(G)$, is the minimum d such that one can assign each vertex a unique vector in $\mathbb{Z}^d$ and two vertices are adjacent in G if and only if the two corresponding vectors differ in all coordinates. In other words, $\mathrm{pd}(G)$ is the minimum d such that there exists some injection $f : V(G) \to \mathbb{Z}^d$ so that u, v are adjacent in G if and only if $\mathrm{dist}(f(u), f(v)) = d$.

The definition and results of the function $f(n, t, X)$ suggest the following variant of the Prague dimension. Given a graph G, the *threshold Prague dimension*, $\mathrm{tpd}(G)$, is the minimum t such that there exists some $d \in \mathbb{N}$ and some $f : V(G) \to \mathbb{Z}^d$ so that u, v are adjacent in G if and only if $f(u)$ and $f(v)$ differ in at least t coordinates, that is, $\mathrm{dist}(f(u), f(v)) \ge t$. By definition, $\mathrm{tpd}(G) \le \mathrm{pd}(G)$. In this section, we list and compare some properties of these two dimensions.

First, $\mathrm{tpd}(G(n, 1/2)) = \Theta(n/\log n)$ with high probability. The upper bound holds as $\mathrm{pd}(G(n, 1/2)) = \Theta(n/\log n)$ with high probability by [GPW23]. For the lower bound, let $\mathcal{G}$ be the set of all graphs with vertex set $[n]$ whose complement has diameter 2. It is well known that $|\mathcal{G}| = (1 - o(1))2^{\binom{n}{2}}$ (see for example [Bol01]). We will compare the number of "essentially distinct" mappings $f : [n] \to \mathbb{Z}^d$ (that realize $\mathrm{tpd}(G)$ for some $G \in \mathcal{G}$) to $|\mathcal{G}|$. Given any graph $G \in \mathcal{G}$ and any $f : V(G) \to \mathbb{Z}^d$ that realizes $t := \mathrm{tpd}(G)$, without loss of generality, we can assume that $f(u) \in [n]^d$ for every vertex u and that $f(1)$ is the all-ones string. Knowing that $\mathrm{dist}(f(u), f(v)) < t$ for u, v not adjacent in G and that the diameter of the complement of G is 2, it follows straightforwardly that $\mathrm{dist}(f(1), f(u)) < 2t$ for all $u \in V(G)$. Define $I_u := \{k \in [d] : f(1)_k \ne f(u)_k\}$ for each $u \in [n]$ and $I := \bigcup_{u\in[n]} I_u$. We have $|I_u| < 2t$ and hence, $|I| < 2tn$. Then, $f(u)_k = f(1)_k = 1$ for every $u \in [n]$ and $k \in [d] \setminus I$. Thus, we can assume $d = 2tn$ without loss of generality. By the former discussion, we can specify any graph $G \in \mathcal{G}$ by a spanning tree T in its complement and the corresponding lists of length $d = 2tn$ so that $f(1)$ is all-ones, $f(u) \in [n]^{2tn}$ for all $u \in [n]$ and $\operatorname{dist}(f(u), f(v)) < t$ for all $u, v \in E(T)$. If u is the parent of v in T and $f(u)$ has been fixed, there are at most $\binom{2tn}{<t} n^t$ choices for $f(v)$. Thus, the number of such functions f is at most $n^{n-2}(\binom{2tn}{<t} n^t)^{n-1} = n^{O(tn)}$ (here n^{n-2} is the number of spanning trees in K_n). If $\operatorname{tpd}(G(n, 1/2)) \le t$ with high probability, then $n^{O(tn)} > |\mathcal{G}| - o(2^{\binom{n}{2}}) = (1 - o(1))2^{\binom{n}{2}}$, i.e. $t = \Omega(n/\log n)$.

Second, if $u_1, u_2, \ldots, u_s$ and $v_1, v_2, \ldots, v_s$ are two sequences of vertices in G such that u_i, v_j are adjacent in G if and only if $i = j$, i.e., the edges (u_i, v_i) form an induced matching in G, then $\operatorname{tpd}(G) \ge \log_2 s$ by Theorem 1.4. Indeed, any $f : V(G) \to \mathbb{Z}^d$ realizing $\operatorname{tpd}(G)$ satisfies $\operatorname{dist}(f(u_i), f(v_i)) \ge \operatorname{tpd}(G)$ for all $i \in [s]$ and $\operatorname{dist}(f(u_i), f(v_j)) < \operatorname{tpd}(G)$ for all distinct $i, j \in [s]$. This argument has been widely used to give lower bounds for $\operatorname{pd}(G)$ for various graphs G. For example, let us consider graphs on n vertices such that the minimum degree is at least one while the maximum degree is Δ. This includes a lot of basic graphs like perfect matchings, cycles, paths, etc. The first author [Alo86] showed that the Prague dimension for these graphs is at least $\log_2 \frac{n}{\Delta} - 2$ because they contain an induced matching of size at least $\frac{n}{4\Delta}$. Now, Theorem 1.4 shows the same bound also holds for the threshold Prague dimension. To compare, we note that Eaton and Rödl [ER96] showed that the Prague dimension (and thus the threshold Prague dimension) for these graphs is at most $O(\Delta \log_2 n)$.

Third, the threshold Prague dimension can be much smaller than the Prague dimension. For example, it is known that $\operatorname{pd}(K_n + K_1) = n$ (see [LNP80]), where $K_n + K_1$ is the vertex disjoint union of a clique of size n and an isolated vertex. However, by mapping the vertices of K_n to the standard orthonormal basis of $\mathcal{R}^n$ and that of K_1 to the all-zeros vector, we observe that $\operatorname{tpd}(K_n + K_1) \le 2$. A more interesting example is the Kneser graph: for $n \ge k$, the Kneser graph $K(n, k)$ is the graph whose vertices are all the k-element subsets of $[n]$ and whose edges are pairs of disjoint subsets. When $1 \le k \le n/2$, it is known that $\log_2 \log_2 \frac{n}{k-1} \le \operatorname{pd}(K(n, k)) \le C_k \log_2 \log_2 n$ for some constant C_k; see [Für00]. For the threshold Prague dimension, define $f : \binom{[n]}{k} \to \{0, 1\}^n$ by mapping each vertex in $K(n, k)$ to the indicator vector of length n of the corresponding subset. Then, for two adjacent vertices u, v, (where the corresponding two subsets are disjoint), $\operatorname{dist}(f(u), f(v)) = 2k$. For two non-adjacent vertices u, v, the two subsets intersect and $\operatorname{dist}(f(u), f(v)) \le 2(k - 1) < 2k - 1$. This shows $\operatorname{tpd}(K(n, k)) \le 2k - 1$. In addition, $K(2k, k)$ is an induced matching of size $\frac{1}{2}\binom{2k}{k}$, so $\operatorname{tpd}(K(2k, k)) \ge \log_2 \frac{1}{2}\binom{2k}{k} = 2k - O(\log_2 k)$. Knowing that $K(n, k)$ contains $K(2k, k)$ as an induced subgraph, we have $\operatorname{tpd}(K(n, k)) \ge \operatorname{tpd}(K(2k, k)) = 2k - O(\log_2 k)$. Thus, $\operatorname{tpd}(K(n, k))$ is asymptotically $2k$. This holds independently of n, very different from the behavior of $\operatorname{pd}(K(n, k))$.

Finally, it would be also interesting to determine the maximum possible threshold Prague dimension for an n-vertex graph G. For the ordinary Prague dimension, this was done by Lovász, Nešetšil and Pultr [LNP80], who showed that $\operatorname{pd}(G) \le n-1$ and $\operatorname{pd}(G) = n-1$ if and only if $G = K_{n-1}+K_1$ (when $n \ge 5$). As we already mentioned above, $K_{n-1} + K_1$ is not a good candidate for maximizing $\operatorname{tpd}(G)$ since $\operatorname{tpd}(K_{n-1} + K_1) \le 2$. Another natural graph to consider is $K_m + K_m$ when $n = 2m$. We claim that $\operatorname{tpd}(K_m+K_m) = \operatorname{pd}(K_m+K_m) = m$. Let the vertex sets of the two cliques be $U = \{u_1, \ldots, u_m\}$ and $V = \{v_1, \ldots, v_m\}$. For the upper bound, assign to u_i an all-is string of length m, and to v_i a string s of length m starting from i in which $s_k = s_{k-1} + 1 \mod m$ for all k. For the lower bound of $\operatorname{tpd}(K_m + K_m)$, suppose $f : U \cup V \to \mathbb{Z}^d$ (for some d) realizes $t := \operatorname{tpd}(K_m + K_m)$. Let

$$C_1 := \sum_{1 \le i < j \le m} \operatorname{dist}(f(u_i), f(u_j)) + \sum_{1 \le i < j \le m} \operatorname{dist}(f(v_i), f(v_j)), \quad C_2 := \sum_{i,j=1}^{m} \operatorname{dist}(f(u_i), f(v_j)).$$

We consider $C_1 - C_2$. Fix $k \in [d]$. For each $a \in \mathbb{Z}$, let s_a be the number of $i \in [m]$ such that $f(u_i)_k = a$, and t_a be the number of $i \in [m]$ such that $f(v_i)_k = a$. The contribution to $C_1 - C_2$ from the kth coordinates is given by

$$\left(\binom{m}{2} - \sum_a \binom{s_a}{2}\right) + \left(\binom{m}{2} - \sum_a \binom{t_a}{2}\right) - \left(m^2 - \sum_a s_a t_a\right) = \sum_a s_a t_a - \frac{s_a^2 + t_a^2}{2} \le 0.$$

Summing over all $k \in [d]$, we know $C_1 \le C_2$. But then, $2\binom{m}{2} \cdot t \le C_1 \le C_2 \le m^2 \cdot (t-1)$, showing $\mathrm{tpd}(K_m + K_m) = t \ge m$, as claimed. It might be the case that $\mathrm{tpd}(G) \le \lceil \frac{n}{2} \rceil$ for all n-vertex graphs G.

5 Concluding remarks and open problems

In Theorem 2.3 we showed there are at most 2^{t+1} pairs of (a_i, b_i) such that $\mathrm{dist}(a_i, b_i) \ge t+1$ for all i and $\mathrm{dist}(a_i, b_j) \le t$ for all $i \ne j$. Consider any $t \ge 1$, the nontrivial case. Notice that $\frac{V_{t+s,s}}{2^{t+s}} \ge \frac{V_{t+3,3}}{2^{t+3}} = \frac{t+4}{2^{t+3}} > 2^{-t-1}$ for all $s \ge 3$. Therefore, Eq. (2) indicates that in the extremal case when there are 2^{t+1} such pairs, it must be that $\mathrm{dist}(a_i, b_i) \in \{t+1, t+2\}$ for all i. By taking $a_i \in \{0,1\}^{t+1}$ and $b_i = \overline{a_i}$, we construct 2^{t+1} such pairs with $\mathrm{dist}(a_i, b_i) = t+1$. Also by taking $a_i \in \{0,1\}^{t+2}$ with an even number of 1s and $b_i = \overline{a_i}$, we construct 2^{t+2} pairs with $\mathrm{dist}(a_i, b_i) = t+2$. Thus, $\mathrm{dist}(a_i, b_i) = t+1$ and $\mathrm{dist}(a_i, b_i) = t+2$ are both possible in the extremal case. It would be interesting to have a further characterization of the extremal cases.

In the realm of the set-pair inequalities, the skew version also plays an important role; see [Lov77, Lov79]. Given t and X, what is the largest m such that there exist $n \ge t+1$ and m pairs $a_i, b_i \in X^n, 1 \le i \le m$ so that $\mathrm{dist}(a_i, b_i) \ge t+1$ for all $i \in [m]$ and $\mathrm{dist}(a_i, a_j) \le t$ for all $1 \le i < j \le m$? We suspect the answer is also 2^{t+1}, and it would be interesting to try to adapt the dimension argument in order to prove it.

In [Für84], Füredi showed the set-pair inequality via the the following vector space generalization. If $A_1, A_2, \ldots, A_m$ are a-dimensional and $B_1, B_2, \ldots, B_m$ are b-dimensional linear subspaces of $\mathbb{R}^n$ such that $\dim(A_i \cap B_j) \le k$ if and only if $i = j$, then $m \le \binom{a+b-2k}{a-k}$. We wonder if there is a natural generalization of Theorem 1.4 or even Theorem 2.3 to vector spaces.

It will be interesting to study the threshold Prague dimension of graphs further. In particular, it will be nice to determine or estimate the maximum possible value of this invariant for a graph with n vertices and maximum degree Δ. In the case of Prague dimension, Eaton and Rödl [ER96] showed the maximum possible dimension of a graph with n vertices and maximum degree Δ is at most $O(\Delta \log n)$ and at least $\Omega(\frac{\Delta \log n}{\log \Delta + \log\log n})$.

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